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The systematic corrections to the photographic magnitudes of the C.P.D. to reduce them to the visual magnitudes of Gould's Zone Catalogue, which are given by Kapteyn (*Annals of the Cape Observatory*, vol. v. p. 32), exhibit a similar discrepancy between the scales of the photographic and visual systems to that which is here disclosed. It will be remembered that Seeliger, in his discussion of the B.D. results, referred to above, found the value of the ratio in question to lie between 3 and 4 for the brighter stars. Seeliger's researches show that the ratio differs in different parts of the sky, and becomes exceptionally large in the galaxy itself, agreeing in this respect with the results for the relative values of the ratios given in columns I. and II. of Table III.

The publication of the authors' further discussion of the system of photographic magnitudes of the C.P.D. will be awaited with interest.

Note on the Principle of the Arithmetic Mean.

By H. C. Plummer, M.A.

1. In this note it is proposed to make a brief comparative study of certain lines of argument which are connected with the establishment of the arithmetic mean of a series of direct observations of a single quantity as the most plausible value to adopt. It has been suggested (by Encke, for example) that this position has been assigned to the arithmetic mean by experience, and even that the principle has an axiomatic character. This does not seem by any means to be the case. The arithmetic mean has always been, and probably always will be, adopted because it is by far the easiest function of the observations to calculate.* Also, although demonstrations of the principle may be wanting in rigour, its position has not been assailed by a rival more formidable in this respect or more acceptable on general grounds. If it be granted that positive and negative errors are equally likely to the same amount, it is obvious—and the formal remark has been made by Ellis †—that the arithmetic mean coincides with the true value in the limit when the number of observations is infinite. For if a is the true value, $x_1, x_2, \ldots$ the n observed quantities, and $e_1, e_2, \ldots$ the errors, then

$$\Sigma(x-a) = \Sigma e,$$

so that

$$a = \frac{\Sigma x}{n} - \frac{\Sigma e}{n} = \frac{\Sigma x}{n}.$$

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* With the exception of the Median, advocated by Edgeworth (*Camb. Phil. Trans.*, 1889, and *Phil. Mag.*, 1887), and by Estienne in his *Etude sur les erreurs d'observation*, Paris, 1890.

† *Camb. Phil. Trans.* vol. viii. p. 205 (1844).

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But the axiom assumed cannot be supposed to hold generally. To consider a very simple example, let observations be made of the position of a point relatively to a scale. If the graduations are equidistant, the axiom will probably hold. If the scale is unequally divided, the axiom seems decidedly improbable. A proof of the principle of the arithmetic mean in the general sense seems therefore to be out of the question. It appears more philosophical to regard the hypotheses on which it has been sought to establish the principle not so much in the nature of axioms as of definitions in the simplest form of the properties which the observations must possess if the principle is to hold.

2. Two preliminary hypotheses which call for little remark may be given. The first is

Hypothesis I.—The function to be adopted is a symmetrical function of the observed quantities.

This must be admitted *a priori*, and must be allowed to stand *a posteriori* if the observations actually made are considered equally plausible. The second is

Hypothesis II.—The function is homogeneous and of the first degree.

This is necessary if the observations are to be allowed the first element of generality. For otherwise the result would not be independent of the unit of measurement.

3. Any function which satisfies these hypotheses, and which becomes identical with the common value of the observed quantities when these coincide, may be called *a* mean value. By the use of Taylor's Theorem De Morgan* expanded such a function in terms of an arbitrarily chosen magnitude and its deviations from the observed quantities. The terms of the first order reduce to the arithmetic mean of the observations, and he thus showed that every mean value consists of the arithmetic mean together with an aggregate of terms with regard to which we have no *a priori* knowledge. Ferrero † has followed the same order of thought, but has replaced the arbitrary magnitude by the arithmetic mean. He then pointed out that the coefficient of terms of the second order in the deviations in general bears a ratio to the reciprocal of n, the number of observations, which does not become infinite with n. He therefore concluded that every mean value approximates more closely to the arithmetic mean as the number of observations is increased and their accordance is improved.

4. A third hypothesis may now be added to the preceding. This is

Hypothesis III.—If all the measures are given any increment a, the most probable value of the quantity sought receives the same increment a.

In other words, the result is independent of the origin of

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* *Camb. Phil. Trans.* vol. x. p. 416 (1864).

† *Esposizione del Metodo dei Minimi Quadrati*, p. 11 (1876).

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measurement. If the most probable value be the function ϕ of the observed quantities $x_1, x_2, \ldots x_n$ the analytical expression of the third hypothesis is obtained at once by making the increment a infinitesimal in this form :

$$\frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_2} + \cdots + \frac{\partial\phi}{\partial x_n} = 1 \quad \ldots \quad \ldots \quad 1)$$

of which the integral is

$$\begin{aligned}\phi &= \frac{1}{n}(x_1 + x_2 + \ldots + x_n) + \mathrm{F}(x_2 - x_1, x_3 - x_1, \ldots, x_n - x_1) \quad (2)\\ &= \mathrm{M} + \mathrm{F}\end{aligned}$$

where M is the arithmetic mean and F is a function of the differences of the observed quantities which satisfies hypotheses I and II. It must be noticed that there is no necessity to suppose that F is a rational and integral function. Otherwise F would vanish by virtue of hypothesis II. This point will be examined further in § 8.

5. The third hypothesis might be replaced by one expressed in a totally different form and yet equivalent to it—

Hypothesis III (*a*).—The law of facility of an error e can be expressed by a function of e.

Let $\psi(e)$ be the function to which the probability of an error e is proportional. If, then, z is the most probable result, the product

$$\psi(z - x_1)\,\psi(z - x_2)\ldots\psi(z - x_n)$$

is a maximum. Therefore

$$\frac{\psi'(z - x_1)}{\psi(z - x_1)} + \frac{\psi'(z - x_2)}{\psi(z - x_2)} + \cdots + \frac{\psi'(z - x_n)}{\psi(z - x_n)} = 0.$$

Now the form of this equation which z must satisfy shows immediately that z satisfies the conditions of hypothesis III. Hypotheses III and III (*a*) are therefore equivalent, and the latter imposes on z the form assigned to ϕ in (2). The geometric mean of the observed quantities, for example, is inconsistent with any law of facility. This has been remarked by Bertrand,[*] who deduced equation (1) from the fact that the functional determinant of

$$\phi - x_1,\ \phi - x_2,\ \ldots,\ \phi - x_n$$

vanishes. This argument seems rather to obscure the real simplicity of the result.

6. Bertrand [†] has also pointed out that the adoption of precise and independent definitions of the terms *precision* and *weight* suffices to restrict the form of the facility function. Let

[*] *Calcul des Probabilités*, p. 182 (1889).

[†] *Loc. cit.* p. 210.

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the functions for two different series of observations be $\psi_1(e)$ and $\psi_2(e)$. Let the ratios of the precisions be h, and the ratio of the weights k, in the two series. The definitions of the terms are such as to imply that the ratios

$$\frac{\psi_1(he)}{\psi_2(e)} \text{ and } \frac{\psi_1(e)^k}{\psi_2(e)}$$

are constant. Accordingly

$$\psi_1(he) = G \, . \, \psi_1(e)^k$$

where G is independent of e. The solution of this functional equation is

$$\psi_1(e) = C \quad \exp.(-ae^\mu)$$

wherein

$$k = h^\mu, \; G = C^{1-k}$$

When $\mu = 2$ the solution gives the orthodox law of facility for which $k = h^2$. But any even values of μ would be equally consistent with definitions of precision and weight based respectively on the idea of equal probability of proportional errors in two systems of observations, and on the idea of the equivalence of a measure in one system to several identical measures in another.

7. It is not intended in this note to consider the form of the facility function as such. But with the general form found in § 6, the most probable value of the quantity sought from a series of observations must make

$$(z-x_1)^\mu+(z-x_2)^\mu+ \, . \, . \, . \, +(z-x_n)^\mu$$

a minimum, so that

$$(z-x_1)^{\mu-1}+(z-x_2)^{\mu-1}+ \, . \, . \, . \, +(z-x_n)^{\mu-1} = 0.$$

This equation, which clearly, since $\mu-1$ is odd, has one and only one real solution, may be written in the form

$$(s_0, s_1, \ldots, s_{\mu-1})(z, -1)^{\mu-1} = 0,$$

where

$$s_r = x_1^r+x_2^r+ \, . \, . \, . \, +x_n^r.$$

In order to obtain a concrete example, let the case be considered in which $\mu = 4$. In the equation

$$(s_0, s_1, s_2, s_3)(z, -1)^3$$

let the roots be diminished by $M = s_1/n$. The result is the transformed equation

$$s_0y^3+3y\left(s_2-\frac{s_1^2}{s_0}\right)-s_3+\frac{3s_1s_2}{s_0}-\frac{2s_1^3}{s_0^2} = 0.$$

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The solution is

$$y=\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}},$$

where α, β are the roots of the quadratic

$$x^2-\left[\frac{s_3}{s_0}-\frac{3s_1s_2}{s_0^2}+\frac{2s_1^3}{s_0^3}\right]x+\left[\frac{s_1^2}{s_0}-s_2\right]^3=0.$$

Hence

$$z=\mathrm{M}+\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}$$

is a function which might be considered the most plausible to adopt as the value of the quantity measured, in the sense that it satisfies the hypotheses so far introduced and is not inherently impossible.

8. The function just found, although not an impossible one nor without a certain interest, has not been constructed for its own sake, but in order to prove by an actual example that hypotheses II and III do not suffice to limit the function of the observed quantities to the arithmetic mean. This is in direct contradiction to the conclusion of Ferrero,* who attempted to prove that the principle of the arithmetic mean was a necessary consequence of these postulates. It has already been noticed in § 4 that F cannot possess a rational and integral form. This suggests that there must be some tacit assumption underlying Ferrero's proof. It is introduced by the expansion of the function in terms of the arithmetic mean and its deviations from the observed quantities. The want of rigour in the proof is due to the assumption that the use of Taylor's Theorem is legitimate. Apparently the existence of functions which cannot be expanded by Taylor's Theorem has been ignored.

9. It is therefore necessary to seek another condition which must be satisfied by the most probable value of the observed quantity. The simplest one seems to be

Hypothesis IV.—The independent measures must be combined in such a way that an error which may exist in one of the measures, as x_1, shall produce the same error in the "value adopted as the most probable" as would be produced by the same error in x_2, x_3 or x_n.

The axiomatic character of this hypothesis does not seem to be obvious, but as an expression of a necessary property of the most probable value it is at least interesting. The condition is at once expressed in analytical form by making the error in $x_1, x_2, \ldots$ or x_n infinitesimal. The equations immediately obtained are

$$\frac{\partial\phi}{\partial x_1}=\frac{\partial\phi}{\partial x_2}=\frac{\partial\phi}{\partial x_3}=\;.\;\;.\;\;.\;=\frac{\partial\phi}{\partial x_n}\quad\cdots\quad\cdots\quad(3)$$

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* *Loc. cit.* p. 40.

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The first equality gives at once

$$\phi = f\,(x_1 + x_2).$$

Hence

$$\frac{\partial\phi}{\partial x_1} = \frac{\partial\phi}{\partial x_2} = \frac{\partial\phi}{\partial(x_1 + x_2)} = \frac{\partial\phi}{\partial x_3},$$

and consequently

$$\phi = f\,(x_1 + x_2 + x_3).$$

Proceeding thus we obtain finally

$$\begin{aligned}\phi &= f\,(x_1 + x_2 + x_3 + \ .\ .\ .\ + x_n)\\ &= \phi\,(\mathrm{M}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)\end{aligned}$$

and if ϕ is a function of invariable form, that form seems to follow most simply by supposing the observed quantities to coincide with a single value a. For then clearly

$$\phi\,(a) = a$$

and

$$\phi = \phi\,(\mathrm{M}) = \mathrm{M} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Thus the principle of the arithmetic mean follows from the hypothesis.

10. Hypothesis IV is due to Stone,* in whose conception its truth seems to have possessed an intuitive character. His method of deducing the equation (4) seems needlessly complicated, and withal not altogether free from objection, as it involved the ex-expansion of the function by Taylor's Theorem. The completion of the proof of the principle in view was effected by the method of induction from the case when $n=2$. In this case, as Glaisher † has pointed out, the truth follows from the theory, though it may be remarked that the "theory" depends on the unproved assumption expressed in Hypothesis III (a). The use of the method of induction seems no more satisfactory and much less simple than the line of argument adopted in deducing equation (5) from equation (4).

11. The last proof of the principle of the arithmetic mean which will be noticed here is the one which is due to Schiaparelli.‡ As a basis of this proof he adopted Hypotheses II, III, and IV. The combination of equations (1) and (3) with the analytical expression of the second hypothesis by means of Euler's Theorem leads at once to the desired result. Schiaparelli arrived at the fourth hypothesis quite independently. It is interesting to notice that these two writers were led to adopt it from quite

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* *Monthly Notices*, vol. xxxiii. p. 570 (1873).

† *Memoirs of the R.A.S.* vol. xxxix. p. 92 (1872).

‡ *Astr. Nachr.* Nr. 2068 (1875).

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different points of view. As has been said, Stone attributed to the condition an *a priori* character. On the other hand, it appears from a later note * that Schiaparelli started with the necessary differential equations and then sought to give them as it were a physical interpretation. Thus Schiaparelli's proof appears frankly artificial, and confirms a remark, made earlier in the present note, that it may be wiser to regard the hypotheses as defining the properties which the observations must possess rather than as axioms on which a rigorous proof of the principle of the arithmetic mean can be founded.

12. I cannot conclude this brief note without expressing my indebtedness to that section of Emanuel Czuber's "Theorie der Beobachtungsfehler" which treats of the principle of the arithmetic mean, both for references and for a connected account of the main lines of earlier discussions of the subject.

University Observatory, Oxford:
1902 *May* 28.

Experimental Reduction of Photographs of Eros for the Determination of the Solar Parallax. Second Paper: *Combination of Results from Mount Hamilton, Minneapolis and Cambridge.* Arthur R. Hinks, M.A.

1. In a recent paper (*Monthly Notices*, 1901 November, vol. lxii. p. 22) I attempted to develop a somewhat new method of comparing with an ephemeris positions of a planet derived from measures upon photographs. Briefly the method was this: Separate heliocentric ephemerides of the planet and of the Earth are expressed in rectangular coordinates referred to a mean ecliptic and equinox, the unit being naturally the mean distance of the Sun from the Earth. From these two simple ephemerides (which arise in the process of computing the usual apparent place ephemeris) it is comparatively easy to compute the position of the planet upon a "standard" plate of given centre; the only transformation of coordinates which is not a mere addition, viz. the change from axes in and perpendicular to the ecliptic to axes in and perpendicular to the plate, is expressed in terms of direction cosines and performed with considerable ease on an arithmometer. In fact, the method offers the advantage that almost throughout the coordinates are rectangular, the transformations linear, and the numerical operations are consequently such as can be performed much more readily with a calculating machine than with logarithms.

2. For a trial of the method I measured and reduced fifty-two exposures of *Eros* made at Cambridge on 1900 November 9 and

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* *Astr. Nachr.* Nr. 2097 (1876).